COEXISTENCE OF COILED SURFACES AND SPANNING SURFACES FOR KNOTS AND LINKS

MAKOTO OZAWA

Abstract. It is a well-known procedure for constructing a torus knot or link that first we prepare an unknotted torus and meridian disks in the complementary solid tori of it, and second smooth the intersections of the boundary of meridian disks uniformly. Then we obtain a torus knot or link on the unknotted torus and its Seifert surface made of meridian disks. In the present paper, we generalize this procedure by a closed fake surface and show that the resultant two surfaces obtained by smoothing triple points uniformly are essential. We also show that a knot obtained by this procedure satisfies the Neuwirth conjecture, and the distance of two boundary slopes for the knot is equal to the number of triple points of the closed fake surface.

1. Introduction

1.1. **The Neuwirth conjecture.** There are not so many geometrical properties satisfied by all non-trivial knots. Any knot bounds a minimal (and hence incompressible) Seifert surface [2], [9], and for any non-trivial knot there exists a properly embedded separating, orientable, incompressible, boundary incompressible and not boundary parallel surface in the exterior of the knot [1]. The following conjecture asserts that any non-trivial knot can be embedded in a closed surface, similarly to the way a torus knot can be embedded in an unknotted torus.

Conjecture 1.1 (Neuwirth Conjecture, [5]). *For any non-trivial knot K in the 3-sphere, there exists a closed surface F containing K non-separatively such that F is essential in the exterior of K.*

Recent results on the Neuwirth conjecture can be seen in [7]. Since the Neuwirth conjecture originated in torus knots, we go back to the construction of torus knots in the next subsection.

1.2. **A procedure for constructing torus knots and links.** The following is a well-known procedure for constructing a torus knot or link [4]. Let T be an unknotted torus in the 3-sphere S^3 which decomposes S^3 into two solid tori V_1 and V_2. Take p mutually disjoint meridian disks D_1 of V_1 and q mutually disjoint meridian disks D_2 of V_2. If we smooth the intersections of ∂D_1 and ∂D_2 uniformly in T, then we can obtain a torus knot or link K of type (p,q). For each point of $\partial D_1 \cap \partial D_2$, we add two triangle regions along this smoothing to $D_1 \cup D_2$, and then we obtain a Seifert surface F_v for K. We remark that by the construction, $\chi(F_v) = |D_1|+|D_2|-|\partial D_1 \cap \partial D_2| = p+q-pq$ and when K is a knot,

2010 *Mathematics Subject Classification.* Primary 57M25; Secondary 57Q35.
Key words and phrases. knot, link, coiled surface, spanning surface, incompressible surface, Neuwirth conjecture.
This work was supported by JSPS KAKENHI Grant Number 23540105.

$g(F_v) = (p-1)(q-1)/2 = g(K)$. We also have cabling annuli $F_h = T \cap E(K)$, where $E(K)$ denotes the exterior of K in S^3. Moreover when K is a knot, we have $\Delta(\partial F_v, \partial F_h) = |\partial D_1 \cap \partial D_2| = pq$, where $\Delta(*,*)$ denotes the distance between two boundary slopes. We note that F_v is orientable and when K is a knot, F_h is connected.

1.3. **From closed fake surfaces to dual surfaces.** We define three subsets of $\mathbb{R}^3$ as below.

(1) $\Sigma_1 = \{(x,y,z) \in \mathbb{R}^3 | z = 0\}$
(2) $\Sigma_2 = \{(x,y,z) \in \mathbb{R}^3 | y = 0, z \geq 0\}$
(3) $\Sigma_3 = \{(x,y,z) \in \mathbb{R}^3 | x = 0, z \leq 0\}$

A finite 2-polyhedron P is called a *closed fake surface* [3] if each of its points has a neighborhood homeomorphic to one of the followings (Figure 1).

Type 1: Σ_1
Type 2: $\Sigma_1 \cup \Sigma_2$
Type 3: $\Sigma_1 \cup \Sigma_2 \cup \Sigma_3$

We will refer to points in closed fake surfaces as points of Type 1, 2 and 3 respectively depending on which of the above three neighborhoods they have. By P', we shall denote the set of points of Type 2 or 3. By P'', we denote the set of points of Type 3. A closed fake surface is *orientable* if each component of $P - P'$ is orientable.

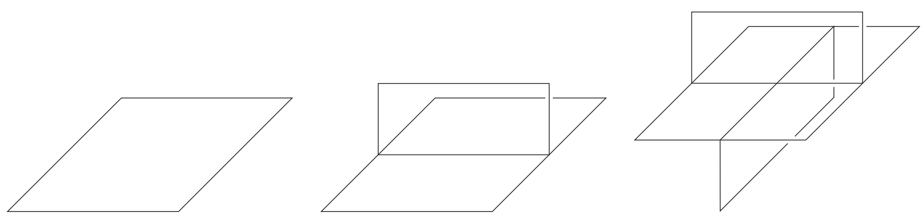

FIGURE 1. Local neighborhoods of a closed fake surface

We say that a closed fake surface P embedded in S^3 has a *vertical-horizontal decomposition* $P = P_v \cup P_h$ if P_h is closed subsurfaces of P which corresponds to $(\Sigma_1, \mathbb{R}^3)$ at each neighborhood of points of Type 2 or 3, and P_v is subsurfaces of P which corresponds to $(\Sigma_2, \mathbb{R}^3)$ or $(\Sigma_3, \mathbb{R}^3)$ at each neighborhood of points of Type 2 or 3. When $P' = \emptyset$, we define $P_h = P$ and $P_v = \emptyset$.

As in the procedure for constructing torus knots and links, for a closed fake surface P with the vertical-horizontal decomposition $P = P_v \cup P_h$, we obtain a knot or link K from P' and the vertical surfaces F_v and the horizontal surfaces F_h from P_v and P_h respectively by smoothing P uniformly as follows. For each neighborhood of a point of Type 3, we add two triangle regions $\{(x,y,z) \in \mathbb{R}^3 | xy \geq 0, |x+y| \leq 1\}$ to P_v. Then we obtain surfaces from P_v and call it the *vertical surfaces* which is denoted by F_v. We note that $\chi(F_v) = \chi(P_v) - |P''|$. The boundary of F_v consists of disjoint simple closed curves in P_h, namely a knot or link, and we denote it by K. The *horizontal surfaces* F_h is the horizontal part P_h of P in $E(K)$. Then we say that F_v and F_h are obtained from P by the $+$-*smoothing*, and that K is obtained from P' by the $+$-*smoothing* (Figure 2). The $-$-*smoothing* of P can be similarly defined, and the results for the $+$-smoothing also hold for the $-$-smoothing. We

note that F_h is always orientable since P_h is closed surfaces in S^3, however, F_v is non-orientable in almost all cases.

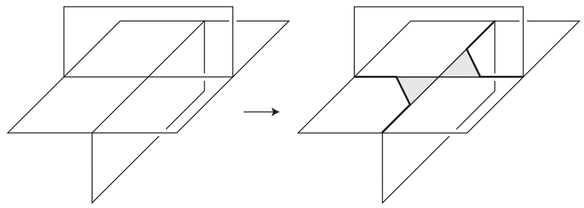

FIGURE 2. $+$-smoothing of a closed fake surface P

1.4. **Definition of essential closed fake surfaces.** Let P be a closed fake surface embedded in the 3-sphere S^3 with the vertical-horizontal decomposition $P = P_v \cup P_h$. A loop l properly embedded in $P - P'$ is *inessential* in P if l bounds a disk δ in $P - P'$, and l is *essential* if it is not inessential. Let $(\alpha, \partial\alpha)$ be an arc properly embedded in $(P_v, P' - P'')$ or $(P_h, P' - P'')$. An arc α is *inessential* in P if there is an arc β in $P' - P''$ such that $\alpha \cup \beta$ bounds a disk in P_v or P_h. Let D be a disk embedded in S^3 such that $D \cap P = \partial D \cap (P - P') = \partial D$. We say that D is a *compressing disk* for P if ∂D is essential in P. We say that D is a *monogon* if $\partial D \subset P_h - P''$ and $|\partial D \cap P'| = 1$. We say that D is a *bigon* if the boundary of D is decomposed into two arcs $\alpha \subset P_v$ and $\beta \subset P_h$ and at least one of α, β is an essential arc in P. A closed fake surface P embedded in S^3 is said to be *essential* if

(1) $S^3 - P$ is irreducible,
(2) P has no compressing disk,
(3) P has no monogon,
(4) P has no bigon, and
(5) P_h has no 2-sphere component.

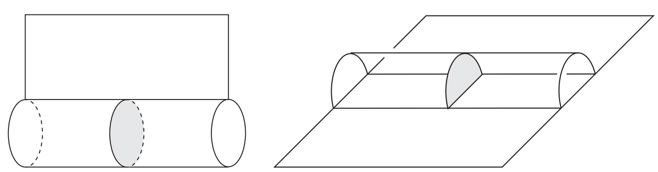

FIGURE 3. A monogon and bigon for $P = P_v \cup P_h$

1.5. **Definition of essential surfaces.** Let K be a knot or link in S^3 and $E(K)$ denote the exterior of K. Let F be a surface properly embedded in $E(K)$, possibly with boundary, except for the 2-sphere or disk, and let i denote the inclusion map $F \to E(K)$. We say that F is *algebraically incompressible* if the induced map $i_* : \pi_1(F) \to \pi_1(E(K))$ is injective, and that F is *algebraically boundary incompressible* if the induced map $i_* : \pi_1(F, \partial F) \to \pi_1(E(K), \partial E(K))$ is injective for every choice of two base points in ∂F.

A disk D embedded in $E(K)$ is a *compressing disk* for F if $D \cap F = \partial D$ and ∂D is an essential loop in F. A disk D embedded in $E(K)$ is a *boundary compressing disk* for F if $D \cap F \subset \partial D$ is an essential arc in F and $D \cap \partial E(K) = \partial D - \mathrm{int}(D \cap F)$. We say that F is *geometrically incompressible* (resp. *geometrically boundary incompressible*) if there exists no compressing disk (resp. boundary compressing disk) for F.

In this paper, we say that surfaces F embedded in $E(K)$ are *geometrically essential* (resp. *algebraically essential*) if each component of F is geometrically (resp. algebraically) incompressible, geometrically (resp. algebraically) boundary incompressible and not boundary parallel. In general, F is algebraically essential if and only if $\partial N(F) \cap E(K)$ is geometrically essential. If F is two-sided in $E(K)$, namely orientable, then F is algebraically essential if and only if it is geometrically essential.

Let K be a knot or link in S^3, and F be closed surfaces embedded in S^3. We say that F is *coiled surfaces* for K if $K \subset F$ and F is geometrically essential in the exterior $E(K)$. We say that a coiled surface F is a *Neuwirth surface* if $F - C$ is connected for each component C of K. Surfaces S embedded in S^3 is *spanning surfaces* for K if $\partial S = K$, and usually S has no closed component.

We remark that any non-trivial, non-splittable knot or link has coiled surfaces since it bounds geometrically incompressible Seifert surfaces F and $\partial N(F)$ gives coiled surfaces. Similarly, if a knot bounds an algebraically incompressible and boundary incompressible spanning surface F, then $\partial N(F)$ gives a Neuwirth surface.

1.6. **Main theorem.**

Theorem 1.2. *Suppose that P is an essential orientable closed fake surface embedded in the 3-sphere S^3 with a vertical-horizontal decomposition $P = P_v \cup P_h$. Let F_v and F_h be the vertical surfaces and the horizontal surfaces respectively obtained from P by the $+$-smoothing, and K be the knot or link obtained from P' by the $+$-smoothing. Then F_v and F_h are algebraically essential in $E(K)$ and K is non-splittable and prime. Moreover when K is a knot, we have that $\Delta(\partial F_v, \partial F_h) = |P''|$ and if F_v is orientable, then F_h is connected.*

We say that a knot or link K is *uniformly twisted* if it can be obtained from P' of an essential orientable closed fake surface P embedded in S^3 with a vertical-horizontal decomposition $P = P_v \cup P_h$ by the $+$-smoothing or $-$-smoothing.

In Theorem 1.2, if F_v is non-orientable, then K can be put on $\partial N(F_v)$ non-separatively. Otherwise, F_h is a Neuwirth surface for K by Theorem 1.2. Hence, we have the following corollary.

Corollary 1.3. *A uniformly twisted knot satisfies the Neuwirth conjecture.*

2. Proof

The proof is straightforward, but the "uniformly smoothing" is a key point.

Proof of Theorem 1.2. First we isotope F_v near points of Type 3 so that F_v intersects F_h in arcs, which form $\{(x,y,z) \| x| \le 1, y = z = 0\}$ in the neighborhoods of that points. Since F_h is orientable and F_v is possibly non-orientable, we need to show that F_h and the (twisted) ∂I-bundle $F_v \tilde{\times} \partial I$ are geometrically incompressible and boundary incompressible in $E(K)$. Then we may assume that in each neighborhood of points of Type 3, $F_h \cap (F_v \tilde{\times} \partial I)$ consists of two arcs.

Suppose that F_h or $F_v \tilde{\times} \partial I$ is compressible in $E(K)$ and let D be a compressing disk for it. Note that D is in the outside of $F_v \tilde{\times} \partial I$ since $F_v \tilde{\times} \partial I$ is incompressible in $F_v \tilde{\times} I$. We take D so that $|D \cap (F_v \cup F_h)|$ is minimal. If $D \cap (F_v \cup F_h) = \emptyset$, then D can be extended to a compressing disk for P. Otherwise, let α be an outermost arc in D and δ be the corresponding outermost disk of D. We extend δ so that $\partial\delta \subset F_v \cup F_h$. Then there are three possibilities, here we note that $\partial\alpha \subset F_v \cap F_h$.

Case 1: α connects two different arcs of $F_v \cap F_h$ which come from distinct points of Type 3.

Case 2: α connects a single arc of $F_v \cap F_h$ which comes from a single point of Type 3, and δ lies in a same side of F_v near the point.

Case 3: α connects a single arc of $F_v \cap F_h$ which comes from a single point of Type 3, and δ lies in both sides of F_v near the point.

In Case 1, δ can not be trivial since there are two arcs of $P' - P''$ in both sides of δ (Figure 4). Here we remember that F_v and F_h are obtained by the $+$-smoothing. Hence δ gives a bigon for P. In Case 2, δ is non-trivial since $|D \cap (F_v \cup F_h)|$ is taken to be minimal. Hence δ also gives a bigon for P. Case 3 does not occur since $\partial\delta$ does not run over another side of F_v since P_v is orientable. Hence F_h and $F_v \tilde{\times} \partial I$ are incompressible in $E(K)$.

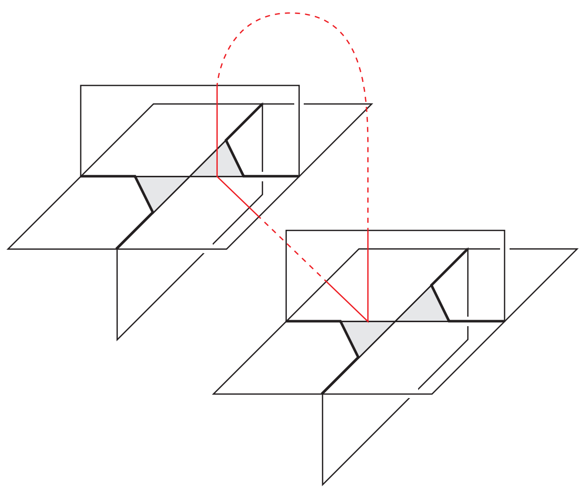

FIGURE 4. The boundary of an outermost disk (Case 1)

In this stage, we can show that K is non-splittable and non-trivial as follows. Let S be an essential 2-sphere in $S^3 - K$. By the incompressibility of F_h, we may assume that S is disjoint from F_h. Moreover, since P_v is incompressible in the complements of F_h, we may assume that S is also disjoint from P_v. Then S bounds a 3-ball in $S^3 - K$ since $S^3 - P$ is irreducible. Hence K is non-splittable. Suppose that K is trivial. Then K is a trivial knot since K is non-splittable. This shows that P_h consists of a single 2-sphere or a single torus since an orientable incompressible surface F_h in a solid torus $E(K)$ is a disk or annulus. In the former case, it contradicts that P_h has no 2-sphere component. In the latter case, F_h is an unknotted torus in S^3 which bounds a solid torus V, and K winds V exactly once.

Then $F_v \cap V$ consists of meridian disks or boundary parallel annuli. If $F_v \cap V$ consists of meridian disks, then V contains a monogon for P, and otherwise V contains a bigon for P. In any case, we have a contradiction. Hence K is non-trivial.

Next, suppose that F_h is boundary compressible in $E(K)$. Since it is well-known that a geometrically incompressible, but geometrically boundary compressible orientable surface in a link exterior is a boundary parallel annulus (c.f. [6, Lemma 2]), there exists a solid torus V bounded by a component T of P_h such that the component of K contained in T winds around V exactly once longitudinally. Since $P_v \cap V$ is incompressible in V, it consists of meridian disks, or boundary compressible annuli. If $P_v \cap V$ consists of meridian disks, then the remaining components of P_v having the boundary in ∂V winds around V exactly once longitudinally. Therefore there exists a monogon for P in V. Otherwise, there exists a bigon for P in V coming from a boundary compressing disk for $P_v \cap V$. Hence F_h is incompressible and boundary incompressible in $E(K)$.

Suppose that $F_v \tilde{\times} \partial I$ is boundary compressible in $E(K)$. Since it is well-known that an algebraically incompressible, but algebraically boundary compressible non-orientable spanning surface for a link is a Möbius band whose boundary is the trivial knot (c.f. [8, Lemma 2.2]), K is the trivial knot. This contradicts that K is non-trivial.

Now we know that both F_h and F_v are not boundary parallel in $E(K)$ since these surfaces are incompressible, boundary incompressible and have integral boundary slopes in $E(K)$. Hence F_v and F_h are algebraically essential in $E(K)$.

In the following, we show that K is prime. Suppose that K is non-prime and let S be a decomposing sphere for K. We may assume that S intersects F_h in two arcs which form a loop l with two points p_1 and p_2 of $K \cap S$. Then l decomposes S into two disks, say D_1 and D_2. By an isotopy, we may assume that ∂D_i does not run over neighborhoods of P'' for $i = 1,\ 2$. Since there is an arc of $D_i \cap P_v$ from a point p_j for $j = 1,\ 2$, there is an arc joining p_1 and p_2 in D_1 or D_2. Thus we may assume without loss of generality that D_1 intersects P_v in a single arc joining $p_1,\ p_2$ and D_2 does not intersect P_v in its interior. Then we have a bigon for P as the subdisk of D_1. Hence K is prime.

Hereafter, we assume that K is a knot. Then F_h consists of a single closed surface. It can be observed that the boundary of F_v and F_h intersects in $|P''|$ points essentially on $\partial N(K)$ since F_v and F_h are obtained by the $+$-smoothing (Figure 5). Thus the distance $\Delta(\partial F_v, \partial F_h)$ is equal to $|P''|$.

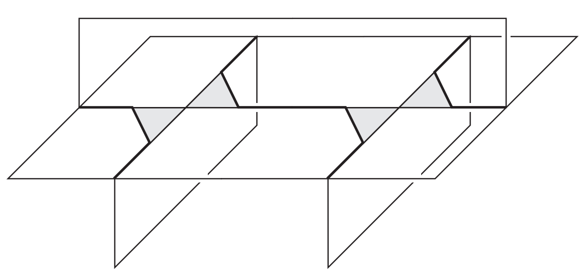

FIGURE 5. $+$-smoothing of a closed fake surface P

Suppose that F_v is orientable. Then K can be oriented by the orientation of F_v. This shows that F_h is connected since F_v and F_h are obtained by the $+$-smoothing (Figure 6). □

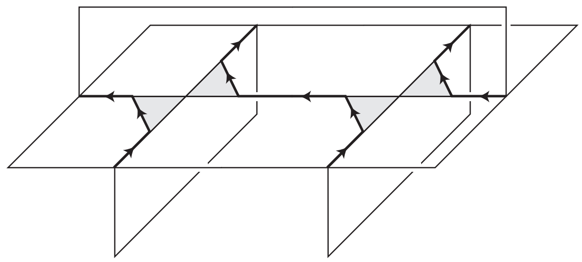

FIGURE 6. An orientation of K induced by F_v

3. EXAMPLE

In this section, we observe that torus links and alternating links are uniformly twisted, which are typical examples for Theorem 1.2.

Let S^2 be a 2-sphere embedded in S^3 and G be a 2-connected graph embedded in S^2 with at least one edge. Then the closed surface $P_h = \partial N(G)$ decomposes S^3 into two handlebodies V_1 and V_2, where V_1 contains G. For each edge G, we take parallel copies of a meridian disk of V_1 which is dual to an edge of G, and for each region of $S^2 - intV_1$, we take parallel copies of a meridian disk of V_2 as the region. Let P_v be a union of these meridian disks. Then we obtain an essential orientable closed fake surface P with the vertical-horizontal decomposition $P = P_v \cup P_h$. Let F_v and F_h be the vertical surfaces and the horizontal surfaces respectively obtained from P by the $+$-smoothing or $-$-smoothing, and K be the knot or link obtained from P' by the $+$-smoothing or $-$-smoothing.

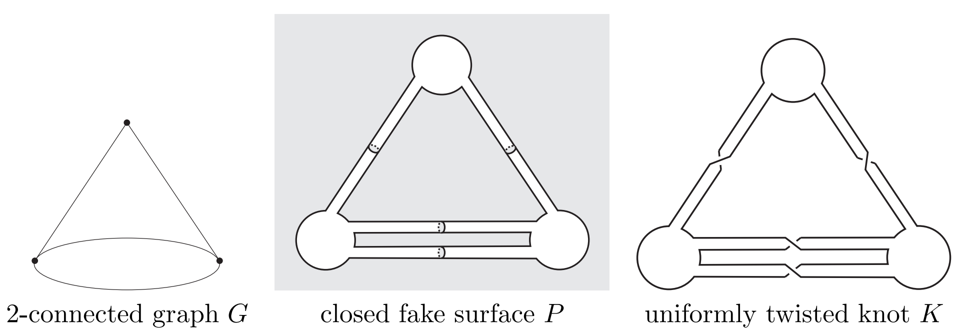

FIGURE 7. The procedure for obtaining a uniformly twisted knot

By the construction, torus knots and links are uniformly twisted. In the above construction, if we take exactly one copy of a meridian disk in V_1 or V_2, then we obtain a prime alternating knot or link as K. In this case, F_v is a checkerboard surface for the alternating diagram of K and F_h is a boundary of a regular neighborhood of

another checkerboard surface. Similarly, we note that generalized alternating knots and links [6] are also uniformly twisted.

4. Problem

We close this paper by some questions and problems.

In the last section, we construct an essential closed fake surface from a trivial closed surface in the 3-sphere. We want to know all essential closed fake surfaces obtained from a given closed surface as their horizontal surface.

Problem 4.1. *Find a construction of an essential closed fake surface from a given closed surface in the 3-sphere.*

The class of uniformly twisted knots and links is somewhat wide, but it has a restriction "uniformly twisted". We want to know how this class is wide.

Problem 4.2. *Does there exist a knot or link which is not uniformly twisted?*

The author think that the answer is yes on this problem since "$\pm$-smoothing" is a strong condition. It seems that the condition in Theorem 1.2 can be weakened.

Problem 4.3. *Find a weaker condition on a closed fake surface which gives still the same conclusion of Theorem 1.2.*

The author would expect that all knots and links can be obtained from a closed fake surface which satisfies the weakend condition.

In general, for a knot satisfying the Neuwirth conjecture, does there exist an essential closed fake surface which gives the knot?

Problem 4.4. *Let K be a knot with a Neuwirth surface F_h, and F_v be an algebraically essential spanning surface for K. Then does there exists an essential closed fake surface P with a vertical-horizontal decomposition $P = P_v \cup P_h$ such that F_h and F_v are obtained from P by the $\pm$-smoothing and K is obtained from P' by the $\pm$-smoothing?*

Finally we extend the Neuwirth conjecture to a link case.

Conjecture 4.5. *For any non-trivial, non-split link L, there exists a closed surface F containing L such that each component of L is non-separating in F and F is essential in $E(L)$.*

Acknowledgements. I would like to thank Yuya Koda for the careful reading and the detailed report.

References

1. M. Culler and P. B. Shalen, *Bounded separating, incompressible surfaces in knot manifolds*, Invent. Math. **75** (1984), 537–545.
2. P. Frankl and L. Pontrjagin, *Ein Knotensatz mit Anwendung auf die Dimensionstheorie*, Math. Ann. **102** (1930), 785–789.
3. H. Ikeda, *Acyclic fake surfaces*, Topology **10** (1971), 9–36.
4. H. C. Lyon, *Torus knots in the complements of links and surfaces*, Michigan Math. J. **27** (1980), 39–46.
5. L. Neuwirth, *Interpolating manifolds for knots in S^3*, Topology **2** (1964), 359–365.
6. M. Ozawa, *Non-triviality of generalized alternating knots*, J. Knot Theory and its Ramifications **15** (2006), 351–360.

7. M. Ozawa and J. H. Rubinstein, *On the Neuwirth conjecture for knots*, to appear in Communications in Analysis and Geometry.
8. M. Ozawa and Y. Tsutsumi, *Totally knotted Seifert surfaces with accidental peripherals*, Proc. Amer. Math. Soc. **131** (2003), 3945–3954.
9. H. Seifert, *Über das Geschlecht von Knotten*, Math. Ann. **110** (1934), 571–592.

Department of Natural Sciences, Faculty of Arts and Sciences, Komazawa University, 1-23-1 Komazawa, Setagaya-ku, Tokyo, 154-8525, Japan

E-mail address: w3c@komazawa-u.ac.jp